

MIDTERM: COMMUTATIVE ALGEBRA I

Date: 5th September 2026

Duration: 3hrs

The Total points is 114 and the maximum you can score is 100 points.

A ring would mean a commutative ring with identity.

1. (2+8+8=18 points) Let R be a ring. Define nil radical of R .
 - (a) R has exactly one prime ideal if and only if every element in R is either a unit or an element of nil radical of R .
 - (b) Let $f \in R$ and let $D(f) = \{\mathfrak{p} \in \text{Spec}(R) \mid f \notin \mathfrak{p}\}$. Then $D(f) = \emptyset$ if and only if $f \in \text{Nil}(R)$.
2. (6+10=16 points) Let $S \subset R$ be a multiplicatively closed subset, and let $S^{-1}R$ be the corresponding ring of fractions.
 - (a) Let I be an ideal of R . Prove that $S^{-1}I = S^{-1}R$ if and only if $I \cap S \neq \emptyset$.
 - (b) For $f \in R$, let R_f be the localization R w.r.t. the set $S = \{1, f, f^2, \dots\}$. Give a bijection from $\text{Spec}(R_f)$ to $D(f)$ defined in the above problem.
3. (8+8=16 points) Exact sequences and Hom functors.
 - (a) Let $0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$ be a sequence of R -modules. Prove that the sequence is split exact if and only if there exists a homomorphism $h : M'' \rightarrow M$ such that $g \circ h = \text{id}_{M''}$.
 - (b) If the sequence splits, show that for any R -module N , the following sequence is exact.
$$0 \rightarrow \text{Hom}_R(N, M') \rightarrow \text{Hom}_R(N, M) \rightarrow \text{Hom}_R(N, M'') \rightarrow 0$$
4. (10 points) Let R be an integral domain with quotient field K . Show that for any torsion R -module T (i.e., every element of T is annihilated by a non-zero element of R), $T \otimes_R K = 0$.
5. (4+10+10=24 points) Define faithfully flat modules.
 - (a) Let $R = k[x]$. Is the module $M = k[x, x^{-1}]$ flat over R ? Is it faithfully flat over R ? Justify your answers completely.
 - (b) Give an example of a projective R -module which is not faithfully flat.
6. (2+8+8=18 points) Let R be a ring and I an ideal of R . Define the subset $V(I)$ of $\text{Spec}(R)$. Show that
 - (a) $V(IJ) = V(I) \cup V(J)$ for all ideals I, J of R .
 - (b) R is an integral domain then $\text{Spec}(R)$ is connected in Zariski topology.
7. (10 points) Let $\phi : M \rightarrow N$ be an R -module homomorphism. Prove that ϕ is surjective if and only if the localized homomorphism $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m} \subset R$.